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Canonical-type connection on almost contact manifolds with B-metric ⋆

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Abstract The canonical-type connection on the almost contact manifolds with B-metric is constructed. It is proved that its torsion is invariant with respect to a subgroup of the general conformal transformations of the almost contact B-metric structure. The basic classes of the considered manifolds are characterized in terms of the torsion of the canonical-type connection.

Keywords almost contact manifold · B-metric · natural connection · canonical connection · conformal transformation · torsion tensor

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Introduction

In differential geometry of manifolds with additional structures, there are important the so-called natural connections, i.e. linear connections with torsion such that the additional structures are parallel with respect to them. There exists a significant interest to these natural connections which have some additional geometric or algebraic properties, for instance about their torsion.

On an almost Hermitian manifold (M,J,g) there exists a unique natural connection ∇^C with a torsion T such that $T(J\cdot,J\cdot)=-T(\cdot,\cdot)$. This connection is known as the canonical Hermitian connection or the Chern connection. An example of the natural Hermitian connection is the first canonical connection of Lichnerowicz ∇^L [15,16]. According to [9], there exists a one-parameter family of canonical Hermitian connections $\nabla^t=t\nabla^C+(1-t)\nabla^L$. The connection ∇^t obtained for $t=-1$ is called the Bismut connection or the KT-connection, which is characterized with a totally skew-symmetric torsion. The latter connection with a

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closed torsion 3-form has applications in type II string theory and in 2-dimensional supersymmetric σ-models [8,26,14]. In [2] and [3] all almost contact metric, almost Hermitian and G_2-structures admitting a connection with totally skew-symmetric torsion tensor are described.

Natural connections of canonical type are considered on the Riemannian almost product manifolds in [10,11,12] and on the almost complex manifolds with Norden metric in [6,4, 24]. The connection in [4] is the so-called B-connection, which is studied in the class of the locally conformal Kählerian manifolds with Norden metric.

In the present paper we consider natural connections (i.e. preserving the structure) of canonical type on the almost contact manifolds with B-metric. These manifolds are the odd-dimensional extension of the almost complex manifolds with Norden metric and the case with indefinite metrics corresponding to the almost contact metric manifolds.

The paper is organized as follows. In Sec. 1 we give some necessary facts about the considered manifolds. In Sec. 2 we define a natural connection of canonical type on an almost contact manifold with B-metric. We determine the class of the considered manifolds where this connection and a known natural connection coincide. In Sec. 3 we consider the group G of the general conformal transformations of the almost contact B-metric structure. We determine the invariant class of the considered manifolds and a tensor invariant of the group G. In Sec. 4 we establish that the torsion of the canonical-type connection is invariant only in the subgroup G_0 of G. We characterize the basic classes of the considered manifolds by the torsion of the canonical-type connection. In Sec. 5 we supply a relevant example.

1 Almost Contact Manifolds with B-metric

Let (M,φ,ξ,η) be an almost contact manifold, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ,ξ,η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ and its dual 1-form η such that the following algebraic relations are satisfied:

$$\varphi\xi = 0, \quad \varphi^2 = -\operatorname{Id}+\eta\otimes\xi, \quad \eta\circ\varphi = 0, \quad \eta(\xi) = 1. \tag{1}$$

Further, let us equip the almost contact manifold (M,φ,ξ,η) with a pseudo-Riemannian metric g of signature $(n,n+1)$ determined by

$$g(\varphi x,\varphi y) = -g(x,y)+\eta(x)\eta(y) \tag{2}$$

for arbitrary elements x, y of the Lie algebra $\mathfrak{X}(M)$ of the smooth vector fields on M. Then (M,φ,ξ,η,g) is called an almost contact manifold with B-metric or an *almost contact B-metric manifold*.

Further, x, y, z will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M is defined by $\tilde{g}(x,y) = g(x,\varphi y)+\eta(x)\eta(y)$. Both metrics g and $\tilde{g}$ are necessarily of signature $(n,n+1)$. The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is also an almost contact B-metric manifold.

Let us remark that the $2n$-dimensional contact distribution $H = \ker(\eta)$, generated by the contact 1-form η, can be considered as the horizontal distribution of the sub-Riemannian manifold M. Then H is endowed with an almost complex structure determined as $\varphi|_H$ – the restriction of φ on H, as well as a Norden metric $g|_H$, i.e. $g|_H(\varphi|_H\cdot,\varphi|_H\cdot) = -g|_H(\cdot,\cdot)$. Moreover, H can be considered as a n-dimensional complex Riemannian manifold with a complex Riemannian metric $g^{\mathbb{C}} = g|_H + i\tilde{g}|_H$ [5].

The structural group of the almost contact B-metric manifolds is $\big(GL(n,\mathbb{C})\cap O(n,n)\big)\times I_1$, i.e. it consists of the real square matrices of order $2n+1$ of the following type

$$\left(\begin{array}{c|c|c} A & B & \vartheta^T \\ \hline -B & A & \vartheta^T \\ \hline \vartheta & \vartheta & 1 \end{array}\right), \qquad \begin{array}{l} A^T A - B^T B = I_n, \\ B^T A + A^T B = O_n, \end{array} \quad A,B\in GL(n;\mathbb{R}),$$

where ϑ and its transpose ϑ^T are the zero row n-vector and the zero column n-vector; I_n and O_n are the unit matrix and the zero matrix of size n, respectively.

1.1 The structural tensor F

The covariant derivatives of φ, ξ, η with respect to the Levi-Civita connection ∇ play a fundamental role in the differential geometry on the almost contact manifolds. The structural tensor F of type (0,3) on (M,φ,ξ,η,g) is defined by

$$F(x,y,z)=g\big((\nabla_x\varphi)y,z\big). \tag{3}$$

It has the following properties:

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi). \tag{4}$$

The relations of $\nabla\xi$ and $\nabla\eta$ with F are:

$$(\nabla_x\eta)y=g(\nabla_x\xi,y)=F(x,\varphi y,\xi).$$

The following 1-forms are associated with F:

$$\theta(z)=g^{ij}F(e_i,e_j,z),\quad \theta^*(z)=g^{ij}F(e_i,\varphi e_j,z),\quad \omega(z)=F(\xi,\xi,z), \tag{5}$$

where g^{ij} are the components of the inverse matrix of g with respect to a basis $\{e_i;\xi\}$ $(i=1,2,\dots,2n)$ of the tangent space T_pM of M at an arbitrary point $p\in M$. Obviously, the equality $\omega(\xi)=0$ and the following relation are always valid:

$$\theta^*\circ\varphi=-\theta\circ\varphi^2. \tag{6}$$

A classification of the almost contact B-metric manifolds with respect to F is given in [7]. This classification includes eleven basic classes $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_{11}$. Their intersection is the special class $\mathcal{F}_0$ determined by the condition $F(x,y,z)=0$. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi=\nabla\xi=\nabla\eta=\nabla g=\nabla\tilde{g}=0$.

Further we use the following characteristic conditions of the basic classes:

$$\begin{array}{rl}
\mathcal{F}_1: & F(x,y,z)=\frac{1}{2n}\big\{g(x,\varphi y)\theta(\varphi z)+g(\varphi x,\varphi y)\theta(\varphi^2 z)\big\}_{(y\leftrightarrow z)};\\
\mathcal{F}_2: & F(\xi,y,z)=F(x,\xi,z)=0,\quad \underset{x,y,z}{\mathfrak{S}} F(x,y,\varphi z)=0,\quad \theta=0;\\
\mathcal{F}_3: & F(\xi,y,z)=F(x,\xi,z)=0,\quad \underset{x,y,z}{\mathfrak{S}} F(x,y,z)=0;\\
\mathcal{F}_4: & F(x,y,z)=-\frac{\theta(\xi)}{2n}\big\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\big\};\\
\mathcal{F}_5: & F(x,y,z)=-\frac{\theta^*(\xi)}{2n}\big\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\big\};\\
\mathcal{F}_{6/7}: & F(x,y,z)=F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\\
 & F(x,y,\xi)=\pm F(y,x,\xi)=-F(\varphi x,\varphi y,\xi),\quad \theta=\theta^*=0;\\
\mathcal{F}_{8/9}: & F(x,y,z)=F(x,y,\xi)\eta(z)+F(x,z,\xi)\eta(y),\\
 & F(x,y,\xi)=\pm F(y,x,\xi)=F(\varphi x,\varphi y,\xi);\\
\mathcal{F}_{10}: & F(x,y,z)=F(\xi,\varphi y,\varphi z)\eta(x);\\
\mathcal{F}_{11}: & F(x,y,z)=\eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\},
\end{array} \tag{7}$$

where (for the sake of brevity) we use the following notations: $\{A(x,y,z)\}_{(x\leftrightarrow y)}$ — instead of $\{A(x,y,z)+A(y,x,z)\}$ for any tensor $A(x,y,z)$; $\mathfrak{S}$ — for the cyclic sum by three arguments; and the former and latter subscripts of $\mathcal{F}_{i/j}$ correspond to upper and down signs plus or minus, respectively.

1.2 The Nijenhuis tensor N

An almost contact structure (φ,ξ,η) on M is called *normal* and respectively (M,φ,ξ,η) is a *normal almost contact manifold* if the corresponding almost complex structure J generated on $M'=M\times\mathbb{R}$ is integrable (i.e. M' is a complex manifold) [25]. The almost contact structure is normal if and only if the Nijenhuis tensor of (φ,ξ,η) is zero [1].

The Nijenhuis tensor N of the almost contact structure is defined by

$$N:=[\varphi,\varphi]+\mathrm{d}\eta\otimes\xi, \tag{8}$$

where $[\varphi,\varphi](x,y)=[\varphi x,\varphi y]+\varphi^2[x,y]-\varphi[\varphi x,y]-\varphi[x,\varphi y]$ and $\mathrm{d}\eta$ is the exterior derivative of the 1-form η. Obviously, N is an antisymmetric tensor, i.e. $N(x,y)=-N(y,x)$. Hence, using $[x,y]=\nabla_x y-\nabla_y x$ and $\mathrm{d}\eta(x,y)=(\nabla_x\eta)y-(\nabla_y\eta)x$, the tensor N has the following form in terms of the covariant derivatives with respect to the Levi-Civita connection ∇:

$$\begin{aligned}N(x,y)=&\left(\nabla_{\varphi x}\varphi\right)y-\varphi\left(\nabla_x\varphi\right)y+\left(\nabla_x\eta\right)y\cdot\xi\\&-\left(\nabla_{\varphi y}\varphi\right)x+\varphi\left(\nabla_y\varphi\right)x-\left(\nabla_y\eta\right)x\cdot\xi.\end{aligned} \tag{9}$$

The corresponding Nijenhuis tensor of type (0,3) on (M,φ,ξ,η,g) is defined by $N(x,y,z)=g(N(x,y),z)$. Then, from (9) and (3) we have

$$N(x,y,z)=\left\{F(\varphi x,y,z)-F(x,y,\varphi z)+F(x,\varphi y,\xi)\eta(z)\right\}_{[x\leftrightarrow y]}, \tag{10}$$

where we use the notation $\{A(x,y,z)\}_{[x\leftrightarrow y]}$ instead of $\{A(x,y,z)-A(y,x,z)\}$ for any tensor $A(x,y,z)$.

Lemma 1 *The Nijenhuis tensor on an arbitrary almost B-metric manifold has the following properties:*

$$\begin{aligned}N(\varphi x,\varphi y,\varphi z)&=-N(\varphi^2x,\varphi^2y,\varphi z)=N(\varphi x,\varphi^2y,\varphi^2z),\\N(\varphi^2x,\varphi y,\varphi z)&=N(\varphi x,\varphi^2y,\varphi z)=-N(\varphi x,\varphi y,\varphi^2z).\end{aligned}$$

Proof Bearing in mind properties (4) of F and relation (10), the equalities in the first line of the statement follow. They imply the equalities in the last line by virtue of (1) and (2). □

Lemma 2 *The class $\mathcal{U}_0=\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$ of the almost contact B-metric manifolds is determined by the condition $N(\varphi\cdot,\varphi\cdot)=0$.*

Proof The statement follows from the following form of the tensor N for each of the basic classes $\mathcal{F}_i$ $(i=1,2,\dots,11)$ of $M=(M,\varphi,\xi,\eta,g)$:

$$\begin{array}{ll}N(x,y)=0, & M\in\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6;\\ N(x,y)=2\left\{\left(\nabla_{\varphi x}\varphi\right)y-\varphi\left(\nabla_x\varphi\right)y\right\}, & M\in\mathcal{F}_3;\\ N(x,y)=4\left(\nabla_x\eta\right)y\cdot\xi, & M\in\mathcal{F}_7;\\ N(x,y)=2\left\{\eta(x)\nabla_y\xi-\eta(y)\nabla_x\xi\right\}, & M\in\mathcal{F}_8\oplus\mathcal{F}_9;\\ N(x,y)=-\eta(x)\varphi\left(\nabla_\xi\varphi\right)y+\eta(y)\varphi\left(\nabla_\xi\varphi\right)x, & M\in\mathcal{F}_{10};\\ N(x,y)=\left\{\eta(x)\omega(\varphi y)-\eta(y)\omega(\varphi x)\right\}, & M\in\mathcal{F}_{11}.\end{array} \tag{11}$$

The calculations are made using (10) and (7). □

2 φ-Canonical Connection

Definition 1 *A linear connection D is called a* natural connection *on the manifold (M,φ,ξ,η,g) if the almost contact structure (φ,ξ,η) and the B-metric g are parallel with respect to D, i.e. $D\varphi = D\xi = D\eta = Dg = 0$.*

As a corollary, the associated metric $\tilde{g}$ is also parallel with respect to a natural connection D on (M,φ,ξ,η,g).

According to [22], a necessary and sufficient condition for a linear connection D to be natural on (M,φ,ξ,η,g) is $D\varphi = Dg = 0$.

If T is the torsion of D, i.e. $T(x,y) = D_x y - D_y x - [x,y]$, then the corresponding tensor of type (0,3) is determined by $T(x,y,z) = g(T(x,y),z)$.

Let us denote the difference between the natural connection D and the Levi-Civita connection ∇ of g by $Q(x,y) = D_x y - \nabla_x y$ and the corresponding tensor of type (0,3) — by $Q(x,y,z) = g(Q(x,y),z)$.

It is easy to establish (see, e.g. [19]) that a linear connection D is a natural connection on an almost contact B-metric manifold if and only if

$$Q(x,y,\varphi z) - Q(x,\varphi y,z) = F(x,y,z), \qquad Q(x,y,z) = -Q(x,z,y). \tag{12}$$

Therefore, according to $T(x,y) = Q(x,y) - Q(y,x)$, we have the equality of the Hayden theorem [13]

$$Q(x,y,z) = \frac{1}{2}\left\{T(x,y,z) - T(y,z,x) + T(z,x,y)\right\}.$$

Definition 2 *A natural connection D is called a* φ-canonical connection *on the manifold (M,φ,ξ,η,g) if the torsion tensor T of D satisfies the following identity:*

$$\begin{aligned}&\big\{T(x,y,z) - T(x,\varphi y,\varphi z) - \eta(x)\left\{T(\xi,y,z) - T(\xi,\varphi y,\varphi z)\right\}\\ &- \eta(y)\left\{T(x,\xi,z) - T(x,z,\xi) - \eta(x)T(z,\xi,\xi)\right\}\big\}_{[y\leftrightarrow z]} = 0.\end{aligned} \tag{13}$$

Let us remark that the restriction of the φ-canonical connection D on (M,φ,ξ,η,g) to the contact distribution H is the unique canonical connection on the corresponding almost complex manifold with Norden metric, studied in [6].

In [21], it is introduced a natural connection on (M,φ,ξ,η,g), defined by

$$\nabla^0_x y = \nabla_x y + Q^0(x,y), \tag{14}$$

where $Q^0(x,y) = \frac{1}{2}\left\{(\nabla_x\varphi)\varphi y + (\nabla_x\eta)y\cdot\xi\right\} - \eta(y)\nabla_x\xi$. Therefore, we have

$$Q^0(x,y,z) = \frac{1}{2}\left\{F(x,\varphi y,z) + \eta(z)F(x,\varphi y,\xi) - 2\eta(y)F(x,\varphi z,\xi)\right\}. \tag{15}$$

The torsion of the φB-connection has the following form

$$T^0(x,y,z) = \frac{1}{2}\left\{F(x,\varphi y,z) + \eta(z)F(x,\varphi y,\xi) + 2\eta(x)F(y,\varphi z,\xi)\right\}_{[x\leftrightarrow y]}. \tag{16}$$

In [23], the connection determined by (14) is called a *φB-connection*. It is studied for some classes of (M,φ,ξ,η,g) in [21,17,18,23]. The restriction of the φB-connection to H coincides with the B-connection on the corresponding almost complex manifold with Norden metric, studied for the class of the locally conformal Kählerian manifolds with Norden metric in [4].

We construct a linear connection ∇' as follows:

$$g(\nabla'_x y,z) = g(\nabla_x y,z) + Q'(x,y,z), \tag{17}$$

where

$$Q'(x,y,z) = Q^0(x,y,z) - \frac{1}{8}\left\{N(\varphi^2 z,\varphi^2 y,\varphi^2 x) + 2N(\varphi z,\varphi y,\xi)\eta(x)\right\}. \tag{18}$$

By direct computations, we check that ∇' satisfies conditions (12) and therefore it is a natural connection on (M,φ,ξ,η,g). Its torsion is

$$T'(x,y,z) = T^0(x,y,z) - \frac{1}{8}\left\{N(\varphi^2 z,\varphi^2 y,\varphi^2 x) + 2N(\varphi z,\varphi y,\xi)\eta(x)\right\}_{[x\leftrightarrow y]}. \tag{19}$$

We verify immediately that T' satisfies (13) and thus ∇', determined by (17) and (18), is a φ-canonical connection on (M,φ,ξ,η,g).

The explicit expression (17), supported by (15) and (10), of the φ-canonical connection by the tensor F implies that the φ-canonical connection is unique.

Immediately we get the following

Proposition 3 *A necessary and sufficient condition for the φ-canonical connection to coincide with the φB-connection is $N(\varphi\cdot,\varphi\cdot) = 0$.*

Thus, Proposition 3 and Lemma 2 imply

Corollary 4 *The φ-canonical connection and the φB-connection coincide on an almost contact B-metric manifold (M,φ,ξ,η,g) if and only if (M,φ,ξ,η,g) is in the class $\mathcal{U}_0$.*

In [22], it is given a classification of the linear connections on the almost contact B-metric manifolds with respect to their torsion tensors T in 11 classes $\mathcal{T}_{ij}$. The characteristic conditions of these basic classes are the following:

$$\begin{array}{rl}
\mathcal{T}_{11/12}: & T(\xi,y,z) = T(x,y,\xi) = 0, \quad T(x,y,z) = -T(\varphi x,\varphi y,z) = \mp T(x,\varphi y,\varphi z);\\
\mathcal{T}_{13}: & T(\xi,y,z) = T(x,y,\xi) = 0, \quad T(x,y,z) - T(\varphi x,\varphi y,z) = \underset{x,y,z}{\mathfrak{S}} T(x,y,z) = 0;\\
\mathcal{T}_{14}: & T(\xi,y,z) = T(x,y,\xi) = 0, \quad T(x,y,z) - T(\varphi x,\varphi y,z) = \underset{x,y,z}{\mathfrak{S}} T(\varphi x,y,z) = 0;\\
\mathcal{T}_{21/22}: & T(x,y,z) = \eta(z)T(\varphi^2 x,\varphi^2 y,\xi), \quad T(x,y,\xi) = \mp T(\varphi x,\varphi y,\xi);\\
\mathcal{T}_{31/32}: & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
 & T(\xi,y,z) = \pm T(\xi,z,y) = -T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{33/34}: & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
 & T(\xi,y,z) = \pm T(\xi,z,y) = T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{41}: & T(x,y,z) = \eta(z)\left\{\eta(y)\hat{t}(x) - \eta(x)\hat{t}(y)\right\}.
\end{array}$$

3 General Contactly Conformal Group G

In this section we consider the group of transformations of the φ-canonical connection generated by the general contactly conformal transformations of the almost contact B-metric structure.

Let (M,φ,ξ,η,g) be an almost contact B-metric manifold. The general contactly conformal transformations of the almost contact B-metric structure are defined by

$$\bar{\xi}=e^{-w}\xi,\quad \bar{\eta}=e^{w}\eta,\quad \bar{g}(x,y)=\alpha g(x,y)+\beta g(x,\varphi y)+(\gamma-\alpha)\eta(x)\eta(y), \tag{20}$$

where $\alpha=e^{2u}\cos 2v$, $\beta=e^{2u}\sin 2v$, $\gamma=e^{2w}$ for differentiable functions u, v, w on M [18]. These transformations form a group denoted by G.

If $w=0$, we obtain the contactly conformal transformations of the B-metric, introduced in [20]. By $v=w=0$, the transformations (20) are reduced to the usual conformal transformations of g.

Let us remark that G can be considered as a contact complex conformal gauge group, i.e. the composition of an almost contact group preserving H and a complex conformal transformation of the complex Riemannian metric $\overline{g^{\mathbb{C}}}=e^{2(u+iv)}g^{\mathbb{C}}$ on H.

Note that the normality condition $N=0$ is not preserved under the action of G. We have

Proposition 5 *The tensor $N(\varphi\cdot,\varphi\cdot)$ is an invariant of the group G on any almost contact B-metric manifold.*

Proof Taking into account (8) and (20), we obtain $\bar{N}=N+(\mathrm{d}w\wedge\eta)\otimes\xi$ and clearly we have $\bar{N}(\varphi x,\varphi y)=N(\varphi x,\varphi y)$. □

According to Lemma 2, we establish the following

Corollary 6 *The class $\mathcal{U}_0$ is closed by the action of the group G.*

Let (M,φ,ξ,η,g) and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be contactly conformally equivalent with respect to a transformation from G. The Levi-Civita connection of $\bar{g}$ is denoted by $\bar{\nabla}$. Using the formula

$$2g(\nabla_x y,z)=xg(y,z)+yg(x,z)-zg(x,y)+g([x,y],z)+g([z,x],y)+g([z,y],x),$$

by straightforward computations we get the following relation between ∇ and $\overline{\nabla}$:

$$\begin{aligned}
&2\left(\alpha^2+\beta^2\right)g\left(\overline{\nabla}_x y-\nabla_x y,z\right)=\\
&=\frac{1}{2}\Big\{-\alpha\beta\left[2F(x,y,\varphi^2 z)-F(\varphi^2 z,x,y)\right]-\beta^2\left[2F(x,y,\varphi z)-F(\varphi z,x,y)\right]\\
&\qquad+\frac{\beta}{\gamma}\left(\alpha^2+\beta^2\right)\left[2F(x,y,\xi)-F(\xi,x,y)\right]\eta(z)\\
&\qquad+2\left(\frac{\alpha}{\gamma}-1\right)\left(\alpha^2+\beta^2\right)F(\varphi^2 x,\varphi y,\xi)\eta(z)\\
&\qquad+2\alpha(\gamma-\alpha)\left[F(x,\varphi z,\xi)+F(\varphi^2 z,\varphi x,\xi)\right]\eta(y)\\
&\qquad-2\beta(\gamma-\alpha)\left[F(x,\varphi^2 z,\xi)-F(\varphi z,\varphi x,\xi)\right]\eta(y)\\
&\qquad-2\left[\alpha\mathrm{d}\alpha(x)+\beta\mathrm{d}\beta(x)\right]g(\varphi y,\varphi z)+2\left[\alpha\mathrm{d}\beta(x)-\beta\mathrm{d}\alpha(x)\right]g(y,\varphi z)\\
&\qquad-\left[\alpha\mathrm{d}\alpha(\varphi^2 z)+\beta\mathrm{d}\alpha(\varphi z)\right]g(\varphi x,\varphi y)+\left[\alpha\mathrm{d}\beta(\varphi^2 z)+\beta\mathrm{d}\beta(\varphi z)\right]g(x,\varphi y)\\
&\qquad+\left[\alpha\mathrm{d}\gamma(\varphi^2 z)+\beta\mathrm{d}\gamma(\varphi z)\right]\eta(x)\eta(y)\\
&\qquad+\frac{1}{\gamma}(\alpha^2+\beta^2)\left\{\mathrm{d}\alpha(\xi)g(\varphi x,\varphi y)-\mathrm{d}\beta(\xi)g(x,\varphi y)\right\}\eta(z)\\
&\qquad+\frac{1}{\gamma}(\alpha^2+\beta^2)\left\{2\mathrm{d}\gamma(x)\eta(y)-\mathrm{d}\gamma(\xi)\eta(x)\eta(y)\right\}\eta(z)\Big\}_{(x\leftrightarrow y)}.
\end{aligned}\tag{21}$$

Using (3) and (21), we obtain the following formula for the transformation by G of the tensor F:

$$\begin{aligned}
2\overline{F}(x,y,z)&=2\alpha F(x,y,z)+\Big\{\beta\left\{F(\varphi y,z,x)-F(y,\varphi z,x)+F(x,\varphi y,\xi)\eta(z)\right\}\\
&+(\gamma-\alpha)\left\{\left[F(x,y,\xi)+F(\varphi y,\varphi x,\xi)\right]\eta(z)+\left[F(y,z,\xi)+F(\varphi z,\varphi y,\xi)\right]\eta(x)\right\}\\
&-\left[\mathrm{d}\alpha(\varphi y)+\mathrm{d}\beta(y)\right]g(\varphi x,\varphi z)-\left[\mathrm{d}\alpha(y)-\mathrm{d}\beta(\varphi y)\right]g(x,\varphi z)\\
&+\eta(x)\eta(y)\mathrm{d}\gamma(\varphi z)\Big\}_{(y\leftrightarrow z)}.
\end{aligned}\tag{22}$$

Proposition 7 *Let the almost contact B-metric manifolds (M,φ,ξ,η,g) and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be contactly conformally equivalent with respect to a transformation from G. Then the corresponding φ-canonical connections $\overline{\nabla}'$ and ∇' as well as their torsions $\overline{T}'$ and T' are related as follows:*

$$\begin{aligned}
\overline{\nabla}'_x y=\nabla'_x y&-\mathrm{d}u(x)\varphi^2 y+\mathrm{d}v(x)\varphi y+\mathrm{d}w(x)\eta(y)\xi\\
&+\frac{1}{2}\big\{\left[\mathrm{d}u(\varphi^2 y)-\mathrm{d}v(\varphi y)\right]\varphi^2 x-\left[\mathrm{d}u(\varphi y)+\mathrm{d}v(\varphi^2 y)\right]\varphi x\\
&\qquad-g(\varphi x,\varphi y)\left[\varphi^2 p-\varphi q\right]+g(x,\varphi y)\left[\varphi p+\varphi^2 q\right]\big\},
\end{aligned}\tag{23}$$

$$\begin{aligned}
\overline{T}'(x,y)=T'(x,y)+\frac{1}{2}\big\{&2\mathrm{d}w(x)\eta(y)\xi+\left[\mathrm{d}u(\varphi^2 x)+\mathrm{d}v(\varphi x)-2\mathrm{d}u(\xi)\eta(x)\right]\varphi^2 y\\
&+\left[\mathrm{d}u(\varphi x)-\mathrm{d}v(\varphi^2 x)+2\mathrm{d}v(\xi)\eta(x)\right]\varphi y\big\}_{[x\leftrightarrow y]},
\end{aligned}\tag{24}$$

where $p = \operatorname{grad} u$, $q = \operatorname{grad} v$.

Proof Taking into account (14), we have the following equality on (M,φ,ξ,η,g):

$$g\left(\nabla^0_x y - \nabla_x y, z\right) = \frac{1}{2}\left\{F(x,\varphi y,z) + F(x,\varphi y,\xi)\eta(z) - 2F(x,\varphi z,\xi)\eta(y)\right\}. \tag{25}$$

Then we can rewrite the corresponding equality on the manifold $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$, which is the image of (M,φ,ξ,η,g) by a transformation from G:

$$\bar{g}\left(\bar{\nabla}^0_x y - \bar{\nabla}_x y, z\right) = \frac{1}{2}\left\{\bar{F}(x,\varphi y,z) + \bar{F}(x,\varphi y,\bar{\xi})\bar{\eta}(z) - 2\bar{F}(x,\varphi z,\bar{\xi})\bar{\eta}(y)\right\}. \tag{26}$$

By virtue of (25), (26), (22) and (21), we get the following formula of the transformation by G of the φB-connection:

$$\begin{aligned}
&g\left(\bar{\nabla}^0_x y - \nabla^0_x y, z\right) = \frac{1}{8}\sin 4v N(\varphi z,\varphi y,\varphi x) - \frac{1}{4}\sin^2 2v N(\varphi^2 z,\varphi^2 y,\varphi^2 x)\\
&-\frac{1}{4}e^{2(w-u)}\sin 2v N(\varphi^2 z,\varphi y,\xi)\eta(x) - \frac{1}{4}\left(1-e^{2(w-u)}\cos 2v\right)N(\varphi z,\varphi y,\xi)\eta(x)\\
&-\mathrm{d}u(x)g(\varphi y,\varphi z) + \mathrm{d}v(x)g(y,\varphi z) + \mathrm{d}w(x)\eta(y)\eta(z)\\
&+\frac{1}{2}\left[\mathrm{d}u(\varphi^2 y) - \mathrm{d}v(\varphi y)\right]g(\varphi x,\varphi z) - \frac{1}{2}\left[\mathrm{d}u(\varphi y) + \mathrm{d}v(\varphi^2 y)\right]g(x,\varphi z)\\
&-\frac{1}{2}\left[\mathrm{d}u(\varphi^2 z) - \mathrm{d}v(\varphi z)\right]g(\varphi x,\varphi y) + \frac{1}{2}\left[\mathrm{d}u(\varphi z) + \mathrm{d}v(\varphi^2 z)\right]g(x,\varphi y).
\end{aligned} \tag{27}$$

From (10), (22), (4) and (20), it follows the formula for the transformation by G of the Nijenhuis tensor:

$$\bar{N}(\varphi x,\varphi y,z) = \alpha N(\varphi x,\varphi y,z) + \beta N(\varphi x,\varphi y,\varphi z) + (\gamma-\alpha)N(\varphi x,\varphi y,\xi)\eta(z). \tag{28}$$

Taking into account (18), (28), (20) and (27), we get (23). As a consequence of (23), the torsions T' and $\bar{T}'$ of ∇' and $\bar{\nabla}'$, respectively, are related as in (24). □

The torsion forms associated with T' of the φ-canonical connection are defined by the following equalities in a similar way of (5):

$$t'(x) = g^{ij}T'(x,e_i,e_j), \quad t'^*(x) = g^{ij}T'(x,e_i,\varphi e_j), \quad \hat{t}'(x) = T'(x,\xi,\xi). \tag{29}$$

Obviously, $\hat{t}(\xi) = 0$ is always valid.

Using (29), (19), (16), (4) and Lemma 1, we obtain that the torsion forms of the φ-canonical connection are expressed by the associated forms with F as follows:

$$t' = \tfrac{1}{2}\left\{\theta^* + \theta^*(\xi)\eta\right\}, \quad t'^* = -\tfrac{1}{2}\left\{\theta + \theta(\xi)\eta\right\}, \quad \hat{t}' = -\omega\circ\varphi. \tag{30}$$

The equality (6) and (30) imply the following relation:

$$t'^*\circ\varphi = -t'\circ\varphi^2. \tag{31}$$

4 General Contactly Conformal Subgroup G_0

Let us consider the subgroup G_0 of G defined by the conditions

$$\mathrm{d}u\circ\varphi^2+\mathrm{d}v\circ\varphi=\mathrm{d}u\circ\varphi-\mathrm{d}v\circ\varphi^2y=\mathrm{d}u(\xi)=\mathrm{d}v(\xi)=\mathrm{d}w\circ\varphi=0. \tag{32}$$

By direct computations, from (7), (20), (22) and (32), we prove that each of the basic classes $\mathcal{F}_i$ $(i=1,2,\dots,11)$ of the almost contact B-metric manifolds is closed by the action of the group G_0. Moreover, G_0 is the largest subgroup of G preserving the 1-forms θ, θ^*, ω and the special class $\mathcal{F}_0$.

Theorem 8 *The torsion of the φ-canonical connection is invariant with respect to the general contactly conformal transformations if and only if these transformations belong to the group G_0.*

Proof Proposition 7 and (32) imply immediately

$$\begin{aligned}\bar{\nabla}'_x y=\nabla'_x y&-\mathrm{d}u(x)\varphi^2y+\mathrm{d}v(x)\varphi y+\mathrm{d}w(\xi)\eta(x)\eta(y)\xi\\&-\mathrm{d}u(y)\varphi^2x+\mathrm{d}v(y)\varphi x+g(\varphi x,\varphi y)p-g(x,\varphi y)q.\end{aligned}\tag{33}$$

The statement follows from (33), or alternatively from (24) and (32). □

Bearing in mind the invariance of $\mathcal{F}_i$ $(i=1,2,\dots,11)$ and T' with respect to the transformations of G_0, we establish that each of the eleven basic classes of the manifolds (M,φ,ξ,η,g) is characterized by the torsion of the φ-canonical connection. Then we give this characterization in the following

Proposition 9 *The basic classes of the almost contact B-metric manifolds are characterized by conditions for the torsion of the φ-canonical connection as follows:*

$$\begin{aligned}
\mathcal{F}_1:\ & T'(x,y)=\tfrac{1}{2n}\left\{t'(\varphi^2x)\varphi^2y-t'(\varphi^2y)\varphi^2x+t'(\varphi x)\varphi y-t'(\varphi y)\varphi x\right\};\\
\mathcal{F}_2:\ & T'(\xi,y)=0,\ \eta\left(T'(x,y)\right)=0,\ T'(x,y)=T'(\varphi x,\varphi y),\ t'=0;\\
\mathcal{F}_3:\ & T'(\xi,y)=0,\ \eta\left(T'(x,y)\right)=0,\ T'(x,y)=\varphi T'(x,\varphi y);\\
\mathcal{F}_4:\ & T'(x,y)=\tfrac{1}{2n}t'^*(\xi)\left\{\eta(y)\varphi x-\eta(x)\varphi y\right\};\\
\mathcal{F}_5:\ & T'(x,y)=\tfrac{1}{2n}t'(\xi)\left\{\eta(y)\varphi^2x-\eta(x)\varphi^2y\right\};\\
\mathcal{F}_6:\ & T'(x,y)=\eta(x)T'(\xi,y)-\eta(y)T'(\xi,x),\ T'(\xi,y,z)=T'(\xi,z,y)=-T'(\xi,\varphi y,\varphi z);\\
\mathcal{F}_{7/8}:\ & T'(x,y)=\eta(x)T'(\xi,y)-\eta(y)T'(\xi,x)+\eta(T'(x,y))\xi,\\
& T'(\xi,y,z)=-T'(\xi,z,y)=\mp T'(\xi,\varphi y,\varphi z)=\tfrac{1}{2}T'(y,z,\xi)=\mp\tfrac{1}{2}T'(\varphi y,\varphi z,\xi);\\
\mathcal{F}_{9/10}:\ & T'(x,y)=\eta(x)T'(\xi,y)-\eta(y)T'(\xi,x),\ T'(\xi,y,z)=\pm T'(\xi,z,y)=T'(\xi,\varphi y,\varphi z);\\
\mathcal{F}_{11}:\ & T'(x,y)=\left\{\hat{t}'(x)\eta(y)-\hat{t}'(y)\eta(x)\right\}\xi.
\end{aligned}$$

Proof According to Proposition 3, Corollary 4, (16) and (19), we have the following form of the torsion of the φ-canonical connection when (M,φ,ξ,η,g) belongs to the classes $\mathcal{F}_i$ $(i\in\{1,2,\dots,11\};i\neq 3,7)$:

$$T'(x,y)=T^0(x,y)=\frac{1}{2}\left\{(\nabla_x\varphi)\varphi y+(\nabla_x\eta)y\cdot\xi+2\eta(x)\nabla_y\xi\right\}_{[x\leftrightarrow y]}.$$

For the classes $\mathcal{F}_3$ and $\mathcal{F}_7$, we use (19) and equalities (11).

Then, using (4), (30), (31) and (7), we obtain the characteristics in the statement. □

According to the classification of the torsion tensors in [22] and Proposition 9, we get the following

Proposition 10 *Let T' be the torsion of the φ-canonical connection on an almost contact B-metric manifold $M=(M,\varphi,\xi,\eta,g)$. The correspondence between the classes $\mathcal{F}_i$ of M and the classes $\mathcal{T}_{jk}$ of T' is given as follows:*

$$
\begin{array}{ll}
M\in\mathcal{F}_1 \Leftrightarrow T'\in\mathcal{T}_{13},\ t'\neq 0; & M\in\mathcal{F}_7 \Leftrightarrow T'\in\mathcal{T}_{21}\oplus\mathcal{T}_{32};\\
M\in\mathcal{F}_2 \Leftrightarrow T'\in\mathcal{T}_{13},\ t'=0; & M\in\mathcal{F}_8 \Leftrightarrow T'\in\mathcal{T}_{22}\oplus\mathcal{T}_{34};\\
M\in\mathcal{F}_3 \Leftrightarrow T'\in\mathcal{T}_{12}; & M\in\mathcal{F}_9 \Leftrightarrow T'\in\mathcal{T}_{33};\\
M\in\mathcal{F}_4 \Leftrightarrow T'\in\mathcal{T}_{31},\ t'=0,\ t'^*\neq 0; & M\in\mathcal{F}_{10} \Leftrightarrow T'\in\mathcal{T}_{34};\\
M\in\mathcal{F}_5 \Leftrightarrow T'\in\mathcal{T}_{31},\ t'\neq 0,\ t'^*=0; & M\in\mathcal{F}_{11} \Leftrightarrow T'\in\mathcal{T}_{41}.\\
M\in\mathcal{F}_6 \Leftrightarrow T'\in\mathcal{T}_{31},\ t'=0,\ t'^*=0; &
\end{array}
$$

5 An example of an almost contact B-metric manifold

In [7], it is given an example of the considered manifolds as follows. Let the vector space $\mathbb{R}^{2n+2}=\left\{\left(u^1,\dots,u^{n+1};v^1,\dots,v^{n+1}\right)\mid u^i,v^i\in\mathbb{R}\right\}$ be considered as a complex Riemannian manifold with the canonical complex structure J and the metric g defined by $g(x,x)=-\delta_{ij}\lambda^i\lambda^j+\delta_{ij}\mu^i\mu^j$ for $x=\lambda^i\frac{\partial}{\partial u^i}+\mu^i\frac{\partial}{\partial v^i}$. Identifying the point $p\in\mathbb{R}^{2n+2}$ with its position vector, it is considered the time-like sphere $S:g(n,n)=-1$ of g in $\mathbb{R}^{2n+2}$, where n is the unit normal to the tangent space T_pS at $p\in S$. It is set $g(n,Jn)=\tan\psi$, $\psi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the almost contact structure is introduced by $\xi=\sin\psi\ n+\cos\psi\ Jn$, $\eta=g(\cdot,\xi)$, $\varphi=J-\eta\otimes J\xi$. It is shown that (S,φ,ξ,η,g) is an almost contact B-metric manifold in the class $\mathcal{F}_4\oplus\mathcal{F}_5$.

Since the φ-canonical connection coincides with the φB-connection on any manifold in $\mathcal{F}_4\oplus\mathcal{F}_5$, according to Corollary 4, then by virtue of (16) we get the torsion tensor and the torsion forms of the φ-canonical connection as follows:

$$
\begin{array}{l}
T'(x,y,z)=\left\{\eta(x)\left\{\cos\psi\ g(y,\varphi z)-\sin\psi\ g(\varphi y,\varphi z)\right\}\right\}_{[x\leftrightarrow y]},\\
t'=2n\sin\psi\ \eta,\qquad t'^*=-2n\cos\psi\ \eta,\qquad \hat{t}'=0.
\end{array}
$$

These equalities are in accordance with Proposition 9. Moreover, it follows that the statement $T'\in\mathcal{T}_{31}$ is valid, which supports Proposition 10.

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